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H_∞ Observer-Based Control Design for Switched Positive Systems Including Interval Uncertainties, Unstable Modes and Time-Varying Delay

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Abstract— *The purpose of this study is to present a practical approach in which the effect of performance degradation and instability factors such as exogenous disturbances, parametric uncertainties, time-varying delay, and unstable modes can reduce to the minimum possible amount in linear switched positive systems. To reduce the effect of the mentioned destructive factors and to strengthen the robust design of switched positive systems, in this paper, instead of using the co-positive Lyapunov function along with the L_1-gain, the quadratic Lyapunov-Krasovskii function utilized along with the L_2-gain, which leads to the design of H_∞ performance. The latter method, especially when there is a requirement to estimate the parameters with the support of the output feedback approach by minimizing the interface parameters, provides the feasibility of designing a more convenient and efficient observer-based controller. The necessary and sufficient conditions for solving the problem concerning the positivity of the system, disturbance attenuation, and parametric uncertainties are expressed by two theories and implemented by the linear matrix inequality technique. The results of this technique's solution include the gains of the controller and observer. Considering that stable and unstable modes are in this system, it is necessary to guarantee the exponential stability of the whole system by the controllers and designing the average dwell-time switching regime. Finally, illustrative examples, including numerical, practical, and comparative, are presented to show the efficiency and performance of different aspects of the proposed approach. The smallness of the mean square error values in the example compared with the output feedback method in linear programming confirms the capabilities of the presented approach. For instance, the mean square error of the system output for the method of this paper is 0.008 and for the compared approach is 0.081.*

Keywords— H_∞ observer-based controller, Interval uncertainty, Lyapunov-Krasovskii function, Switched positive system, Time-varying delay.

1. INTRODUCTION

The contents of this section have been described in three subsections due to the more complete and accurate explanation of the contents. In addition, in this way, each subsection has more coherence in representing the subject matter.

1.1. System description and its applications

Switched systems are a widely used class of hybrid systems, including a finite number of modes (subsystems) or dynamics and a switching signal to govern the switching rules. Significant research attention was paid to these types of systems in the last few decades for two main reasons: first, a notable quantity of components cannot be described by only a single-mode system, and second, a solitary controller may not work precisely for the numerous control targets. Many practical systems are inherently multimodal in the sense that to describe their behavior more accurately, there is no other way than to use switching systems such as electrical power systems [1], power electronics [2], robotics [3], cyber-physical systems [4], and biological systems [5]. The great extent of switched systems and their engagement in many fields has caused these systems to be analyzed and investigated from various aspects. Among the essential and noteworthy reviews that have been accomplished regarding such systems, one can mention stabilization and stability analysis [6], robust control design [7], observation-based design [8], filtering [9], positive method [10], and so on.

A portion of the systems, which are studied and evaluated in the category of switched systems, is positive compartmental systems. Such practical systems are derived from mass and energy balance relations; their state variables intrinsically take positive (non-negative) values [11]. Positive compartmental systems are necessarily positive in the sense that their state trajectories and outputs are always non-negative whenever the initial conditions are non-negative as well [12], while this category of systems can have a switching or non-switching structure [13]. For both switched or non-switched cases, the states instead of the whole space are confined in a cone in the positive orthant. The presented applications in the fields of medication [14], the concentration of matter [15], population of groups [16], heat exchange [17], and epidemiological systems [18], are the most accustomed examples in connection with the positive compartmental systems. Although a Switched Positive System (SPS) is considered a type of switched system, the conditions governing their design differ in many ways. A typical switched system differs from an SPS in these ways: the initial values in a switched system can be positive or negative, but in SPS it must be non-negative, the input disturbance in a switched

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system can assume negative values, but it is customary that the disturbance in SPS is always non-negative, the input, output, delay and disturbance matrices must have non-negative entries, and the state matrix in SPS must be Metzler. The significance of the issue is that when the design is based on the distinctions mentioned above, the system is guaranteed to remain positive in every regard. Among the crucial research regarding positive systems and positive switching, the following can be mentioned: L_1-gain stability analysis based on co-positive Lyapunov function [19] positive observer design [20], positive observer [21], robust H_∞ positive design [22], etc.

In actual conditions, plants often encounter phenomena that change their efficiency and performance. Parametric uncertainty, time delay, and disturbance are the main types of these phenomena. In this study, an attempt is made to propose a design that effectively minimizes the effect of the noted factors on reducing the efficiency and instability of the system at the same time. If the uncertainty falls within a wide range, it should be managed by adaptive control methods, while the limited cases can be handled by other robust methods such as our design. Parametric uncertainty for SPS is classified into two principal types: polytopic and interval. Both cited parametric uncertainties with state feedback controllers and co-positive Lyapunov function have been discussed in [23]. In switching systems, the time-delay is either a constant value or a time-varying function that can be single or multiple. There can also be a distributed time delay. In addition, there can be a combination of these delays [24]. In [25], the stabilization of an SPS with disturbance and the distributed time delay is discussed. Disturbance signal is another source of error (such as parameter uncertainty) in the system, and its attenuating has a crucial influence on improving the system's performance. For the robust design of positive systems against disturbances, L_1-gain performance is used much more than L_2-gain. The reason is that some researchers believe that using the L_1-gain approach is better because it leads to a positive controller and observer design. But using this method is not beneficial for every design, which we will discuss further. In this study, the quadratic Lyapunov function and the L_2-gain performance are consciously utilized to obtain valid and feasible answers and to calculate the controller and observer gains more accurately. Lyapunov-like function for L_2-gain stability analysis has been used in [26], and co-positive type Lyaponuv-Krasovskii function for L_1-gain performance as another has been studied in [27]. According to [28], regarding the relationship between L_2-gain and H_∞ performance, it can be stated that: the L_2-gain method utilizes Lyaponuv-like function because for linear systems L_2-gain can be found exactly and is equivalent to the infinity norm, H_∞, of the transfer function from disturbance to penalty vector.

Switching regimes in constrained switching systems play a decisive role in system stability analysis. An improper switching regime destabilizes the system even if all the subsystems are stable, and under some appropriate circumstances, the switching behavior can stabilize the system with all subsystems unstable [29]. State-dependent and time-dependent two types of constrained switching regimes have been considered in [30]. In this paper, in addition to proving the stability of the design made by the Lyapunov method, it is necessary to prove the stability for the switching regime as well. The switching regime in this work is time-dependent, and its stability is proven by the Average Dwell Time (ADT) method. The design method is such that, as noted in [29], if unstable modes apply in the design conditions, the system will remain stable despite the unstable modes. According to the ADT method number of switches in a finite interval is bounded, and the average time between consecutive switching should not be less than the definite time [31].

The design proposed in this study can be utilized for robust control of all types of linear positive compartmental switched systems in which one or some state variables cannot be measured and should be estimated. Such as pest population groups [32], heat in different parts of the rocket engine, and so many medical affairs like control of the anesthetic drug in the body [33]. With the aid of the presented design, all the non-measurable state variables are estimated with the output feedback method and are employed in the control process. Also, the effect of external disturbances on the system performance will be decreased to the lowest level. The SPS challenges are solved, with the following techniques in brief. The problem of interval parameter uncertainty utilizes an analytical method and Linear Matrix Inequality (LMI). For time-varying delay issue, Lyapunov-Krasovskii function is used. The adverse effect of external disturbance is attenuated by the robust H_∞ design. For the proper operation of the switching regime in stability analysis ADT approach is utilized.

1.2. Relevant works and contributions

In this subsection, the closest works to this research are briefly discussed, and their similarities and differences are expressed. Also, the innovations and contributions presented in this paper are better clarified. To the best of our knowledge, the problem of the H_∞ observer-based control for SPS has not yet been investigated, especially when the factors that generally affect the performance of a control system are considered, such as parametric uncertainties, exogenous disturbance, and time-varying delay. Moreover, in this paper, there is a great emphasis on the feasibility of answers. That is, the answers of the MATLAB solvers are re-checked based on the assumptions of the problem to get definitive correct answers instead of the closest answers. In other terms, by changing design parameters, exact solutions are obtained instead of probably proper solutions.

In [34], for a non-switched delayed positive system with interval uncertainty, a controller and an observer were designed. In that work, since the mentioned system is not switching, the topics related to the stabilization of a switching regime (such as dwell-time switching) have not been expressed. Therefore, it is not possible to adjust the dwell-time to stabilize the system despite the stable and unstable modes [29]. Moreover, the disturbance is not considered for the system [34], and subsequently, H_∞ controller design has no use for it. In our study, the method of the recent paper was inspired, for examining the conditions of positivity of the system with parametric uncertainty, although according to the requirements of the problem, a different solution method and Lyapunov function were employed. Delay-dependent observer-based finite-time control for a non-positive switched system has been discussed in [35]. The method of Lyapunov-like function and ADT switching signal design in the mentioned paper promoted our research. Despite these similarities, our design is primarily different from this article due to the positivity of the system and presence of parameter uncertainty. The nearest study to this paper is the one presented in [36], where the observer-based controller for a non-linear SPS is designed with a sliding mode approach. This research is one of the rare works in SPSs that utilized an observer-based control design. In the recent article, output feedback is employed in the sliding mode structure, and there is no uncertainty. The problem with this design is that due to the usage of the L_1-gain performance and the co-positive Lyapunov function in the disturbance attenuation, a large number of design variables and interface variables have been created, which limits the feasibility of the solution to a great extent. It should be noted that the L_1-gain method is most often unable to obtain feasible solutions for positive switched output feedback systems, due to generating the BMI (Bilinear Matrix Inequality) terms and using more parameters and vectors compared to the L_2-gain method [37]. In [38], a controller was designed based on the co-positive Lyapunov function for both state feedback and output feedback systems and both for discrete time and continuous time cases without the presence of the observer. The weakness of the approach presented in this article is the use of non-linear relations and BMIs, which no solver can solve them. Also, no

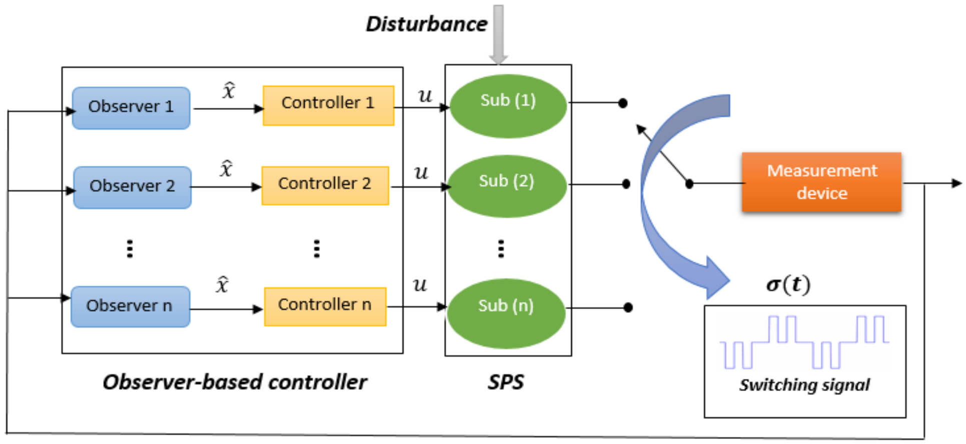

Fig. 1. Closed-loop system operation

example is provided for the output feedback design method. In [39], a robust observer based on the co-positive Lyapunov function was designed for an SPS with time-varying delay, and without parameter uncertainty, disturbance, and controller. In this article, because there is no controller, despite using the co-positive Lyapunov function, the number of parameters (vectors) of the positive interface is not very much. The study presented in [40] includes state-dependent switching law, L_1-gain performance, and tracking control design which, because of its switching regime, is separate from our introduced works. Another paper [41] introduced output feedback for finite-time L_1-gain control, while none of the states converged in its numerical example.

In this study, the aim is to construct H_∞ observer-based controllers for an SPS to guarantee the positive stability of the closed-loop system and overcome uncertainty, disturbance, time-varying delay, and the destructive effect of unstable states. Observer-based control design for SPSs, which requires the use of output feedback, is rarely seen. For the first time, in our approach the Lyapunov-Krasovskii quadratic function and L_2-gain performance are employed. Utilizing this methodology removes unnecessary interface parameters and leads to feasible answers.

The main contributions of this paper are given as follows:

1) Observer-based stabilization of switched system with interval uncertain system and time-varying delay by means of LMIs.
2) Providing sufficient conditions for positive observer-based stabilization and H_∞ stabilization.
3) H_∞ observer-based control design considering exponential stability.
4) Employing the Lyapunov-Krasovskii quadratic function and L_2-gain performance for an SPS in output feedback H_∞ observer-based controller design.
5) Designing a switching regime based on the ADT approach and guaranteeing its stability after the stabilization of unstable subsystems.

1.3. Open problems

Applying the design presented in this paper for the systems with a purely nonlinear model or linear model with a nonlinear term can have various applications. Also, since in the method of this study we need measurement and estimation, in many cases we are met with measurement and process noise. Including noise in the system model and using a suitable Kalman filter for switching systems such as IMM-KF can greatly increase the accuracy of the design. In cases like the proposed problem in [33], an adaptive controller has been used, which can achieve effective responses by using the observer-based control method and employing the reference value, and transferring the equilibrium point of the system. It will be very beneficial to design this model of the system in such a way that it has extended stability with all the modes unstable without the controllers.

1.4. Paper structure

The rest of this paper is organized as follows: In Section 2, the system model and some basic lemmas and definitions are given. Section 3 is devoted to deriving the results on H_∞ stabilization, designing the robust observer-based controller, and comparing it to another approach. Three types of example are given in Section 4 to verify the theoretical results. Finally, The L_2-gain stability method for developing an observer-based controller, along with other advantages of this design, is briefly explained in Section 5.

Nomenclature:

- R^n denotes the n-dimensional real Euclidean space.
- R_+^n is the set of n-dimension vectors whose elements are all positive.
- $R^{n\times k}$ is the set of all real matrices of dimension $(n \times k)$.
- $X \succcurlyeq 0 (\preccurlyeq 0)$ means that all entries of the matrix X are non-negative (non-positive).
- $X \succcurlyeq Y$ means that $X - Y \succcurlyeq 0$.
- $\lambda(X)$ means all eigenvalues of the matrix X.
- $\max_m \lambda(X_m)$ means maximum eigenvalues of matrices $X_1 X_2 \ldots\ X_s$.
- $\|X\|$ denotes the norm $X^T X = \left(x_1^2,\ x_2^2,\ \ldots,\ x_n^2\right)$, where x_i is the ith element of vector $X \in R^n$.
- $[X]_{ij}$ denotes the element located at the i^{th} row and j^{th} column.
- For the matrix X, $X \in [\underline{X}, \overline{X}]$ specifies that $\underline{X} \preccurlyeq X \preccurlyeq \overline{X}$.
- $trace\,(X) = \sum_{i=1}^{n} x_{ii} = \sum_{i=1}^{n} \lambda_i$, is defined to be the sum of elements on the main diagonal and also is the sum of the eigenvalues.
- $\|.\|_{L_2}$ is the L_2 norm of a signal defined as $\|x(t)\|_{L_2} = \left(\int_0^\infty \|x(s)\|^2 ds\right)^{1/2}$.
- The symbol '*' is used widely in linear matrix inequalities which represents the symmetric element of the matrix; the superscript 'T' stands for matrix transpositions.
- $\vartheta\,(X)$ for any matrix $X \epsilon R^{n\times n}$ $\vartheta\,(X) = \max_i\{Re\,(\lambda_i)\}$ indicates its spectral abscissa where $\lambda_1 \ldots \lambda_n$ are eigenvalues of X.
- $\sigma\,(t)$ is the switching signal.
- $V_\sigma\,(t)$ Lyapunov-Krasovskii function.
- J indicates cost function based on L_2-gain stability that should be optimized.
- τ_a represents average dwell-time.
- MSE represents Mean Square Error.

2. Preliminaries and Problem Formulation

In this section, an observer and controller design is given for the following interval switched positive system with a time-varying delay.

$$\begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + A_{1,\sigma(t)}x(t-h(t)) \\ \quad +B_{\sigma(t)}u(t) + D_{1,\sigma(t)}\omega(t) \\ y(t) = C_{\sigma(t)}x(t) \\ x(t) = \varphi(t) \geq 0 \quad t \in [-\overline{h}, 0] \\ z(t) = C_{2,\sigma(t)}x(t) + A_{2,\sigma(t)}x(t-h(t)) \end{cases} \tag{1}$$

where $x(t) \in R^n$ is state vector, $u(t) \in R^m$ is control input, $y(t) \in R^s$ is the measured output, $z(t) \in R^q$ is the controlled output or penalty vector. $\omega(t) \in R^s$ is exogenous disturbance input that satisfies Assumption 1 stated below. $\varphi(t)$ is the continuous vector-valued function specifying the initial state of the system, $\sigma(t) : [0, \infty) \to M = \{1, 2, \ldots, N\}$ is the switching signal, with N is the number of subsystems, and the switching sequence is $\{x_0; (m_0, t_0), (m_1, t_1), \ldots, (m_k, t_k), \ldots, | \ m_k \in M, \quad k = 0, 1, \ldots\}$ which means the $m_k th$ subsystem is activated when $t \in [t_k, t_{k+1})$. $h(t)$ denotes the time-varying delay satisfying Assumption 2 stated below. $(A_m,\ A_{1,m}, B_m, C_m, C_{2,m}, A_{2,m})$ are system matrices. $(D_{1,m})$ is disturbance matrix. For any $m \in M$, the matrices $A_m \in [\underline{A_m}, \overline{A_m}]$, $B_m \in [\underline{B_m}, \overline{B_m}]$, $A_{1,m} \in [\underline{A_{1,m}}, \overline{A_{1,m}}]$ and $C_m \in [\underline{C_m}, \overline{C_m}]$ are unknown constant matrices with known bounds, which means that these system matrices include uncertainties.

Assumption 1. The disturbance signal $\omega(t)$ is time-varying and satisfies:

$$\int_0^\infty \omega(t)^T \omega(t)\, dt < l \ , \quad l \geq 0. \tag{2}$$

Assumption 2. The time-varying delay $h(t)$ satisfies:

$$0 \ \leq h(t) < \overline{h} \ , \qquad \dot{h}(t) \leq d < 1. \tag{3}$$

In the following, some definitions and basic results for switched positive systems are presented.

Definition 1. System (1) is said to be positive if and only if its states and output are nonnegative for every nonnegative initial state.

Definition 2. ([12]) A is a Metzler matrix if the off-diagonal entries of this matrix are non-negative.

Lemma 1. System (1) is positive if and only if for every $m \in M$, A_m are Metzler matrices, and $A_{1,m} \succcurlyeq 0$, $B_m \succcurlyeq 0, C_m \succcurlyeq 0, C_{2,m} \succcurlyeq 0$, $A_{2,m} \succcurlyeq 0$.

Definition 3. ([35]) For any $T \geq t \geq 0$, let $N_\sigma(t, T)$ be the switching number of $\sigma(t)$ over (t, T). If the following inequality holds for $\tau_a \geq 0$ and integer $N_0 \geq 0$, then τ_a is called average dwell time.

$$N_\sigma(t, T) \leq N_0 + \frac{T - t}{\tau_a} \tag{4}$$

Lemma 2. ([42]) For matrices X, $Y \epsilon\ R^{n\times n}$, if X is Metzler and $Y \succcurlyeq X$, then $\vartheta(Y) \geq \vartheta(X)$.

The following state observer is considered to estimate the states of system (1):

$$\dot{\hat{x}}(t) = G_{\sigma(t)}\hat{x}(t) + A_{1,\sigma(t)}\hat{x}(t - h(t)) + L_{\sigma(t)}y(t) \tag{5}$$

where $A_{1,\sigma} \in R^{n\times n}$ is known and $G_\sigma \in R^{n\times n}$ and $L_\sigma \in R^{n\times q}$ to be determined. The output-feedback control law is as follows:

$$u(t) = K_{\sigma(t)}\hat{x}(t) \tag{6}$$

where $K_\sigma \in R^{m\times n}$ is the gain of controller and to be determined.

Remark 1. Stabilization and estimation problems are crucial issues for every control system. The basis of design in this system is based on the estimation of state variables and the formation of the augmented system. As stated in [37], the employ of the co-positive Lyapunov function is unable to provide feasible answers due to two reasons, the production of BMI terms and a large number of intermediate parameters in the design. In addition, compared to the approach presented here, that mentioned method requires more repetition in the recursive process of solving the problem and more sensitivity to the initial values.

How the closed-loop system operates for the model introduced in this paper can be seen in Fig. 1. In this figure, Sub 1, Sub 2, ..., and Sub n represent the modes or subsystems in the model. To guarantee the positivity of the estimated $\hat{x}(t)$, it is required that $G_{\sigma(t)}$ be Metzler and $L_{\sigma(t)}$ be non-negative matrices. By considering the control law (6) and from (1), (5), the closed-loop system is as follows:

$$\begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + A_{1,\sigma(t)}x(t-h(t)) \\ \quad +B_{\sigma(t)}K_{\sigma(t)}\hat{x}(t) + D_{1,\sigma(t)}\omega(t) \\ z(t) = C_{2,\sigma(t)}x(t) + A_{2,\sigma(t)}x(t-h(t)) \\ y(t) = C_{\sigma(t)}x(t) \\ x(t) = \varphi(t) \geq 0 \quad t \in [-\overline{h}, 0] \\ \hat{x}(t) = \nu(t) \geq 0 \quad t \in [-\overline{h}, 0] \end{cases} \tag{7}$$

where $\nu(t)$ is the initial function for state estimation and the signal $e(t) = x(t) - \hat{x}(t)$ is defined as state estimation error. From (5) and (7) we can write:

$$\tilde{x}(t) = \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \tag{8}$$

$$\dot{\tilde{x}}(t) = \begin{bmatrix} (A_\sigma + B_\sigma K_\sigma)x(t) - B_\sigma K_\sigma e(t) + A_{1,\sigma}x(t-h(t)) + D_{1,\sigma}\omega(t) \\ (A_\sigma + B_\sigma K_\sigma - L_\sigma C_\sigma - G_\sigma)x(t) + (G_\sigma - B_\sigma K_\sigma)e(t) + A_{1,\sigma}e(t-h(t)) + D_{1,\sigma}\omega(t) \end{bmatrix} \tag{9}$$

$$\dot{\tilde{x}}(t) = \begin{bmatrix} A_\sigma + B_\sigma K_\sigma & -B_\sigma K_\sigma \\ A_\sigma + B_\sigma K_\sigma - L_\sigma C_\sigma - G_\sigma & G_\sigma - B_\sigma K_\sigma \end{bmatrix}\tilde{x}(t) + \begin{bmatrix} A_{1,\sigma} & 0 \\ 0 & A_{1,\sigma} \end{bmatrix}\tilde{x}(t-h(t)) + \begin{bmatrix} D_{1,\sigma} \\ D_{1,\sigma} \end{bmatrix}\omega(t) \tag{10}$$

where

$$A_{x,\sigma} = \begin{bmatrix} A_\sigma + B_\sigma K_\sigma & -B_\sigma K_\sigma \\ A_\sigma + B_\sigma K_\sigma - L_\sigma C_\sigma - G_\sigma & G_\sigma - B_\sigma K_\sigma \end{bmatrix} \tag{11}$$

$$A_{h,\sigma} = \begin{bmatrix} A_{1,\sigma} & 0 \\ 0 & A_{1,\sigma} \end{bmatrix} \tag{12}$$

$$A_{w,\sigma} = \begin{bmatrix} D_{1,\sigma} \\ D_{1,\sigma} \end{bmatrix} \tag{13}$$

Definition 4. Switched positive system (1) with observer-based control is said to be H_∞ observer-based stabilizable concerning (2), $\gamma > 0$, and the switching signal $\sigma(t)$ if the following conditions are satisfied:

1) Switched positive system (1) with observer-based control is stabilizable.
2) Under zero initial condition $\varphi(t) = 0 \quad \forall t\epsilon[-\overline{h}, 0]$, the output $z(t)$ satisfies:

$$\int_0^\infty z^T(t)\, z(t)\, dt < \ \gamma^2 \int_0^\infty \omega(t)^T \omega(t) dt \tag{14}$$

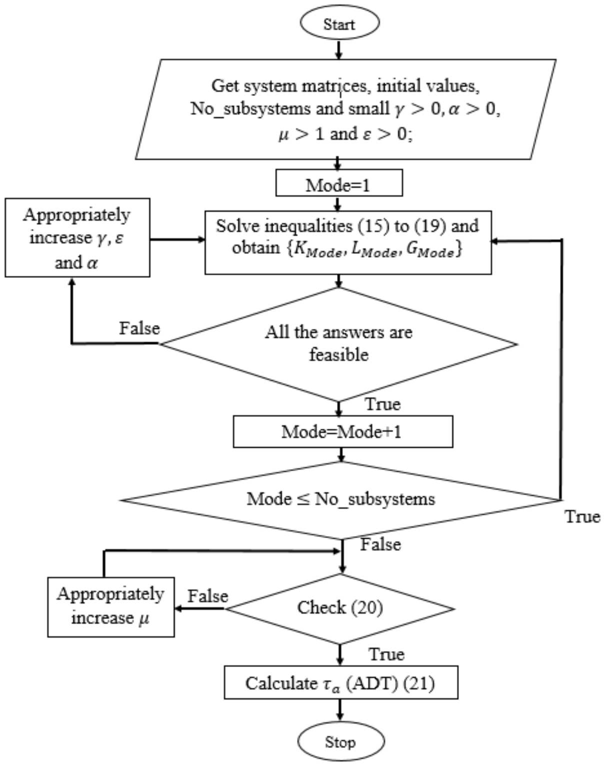

Fig. 2. H_∞ observer-based design flowchart

Remark 2. In Definition 4, γ means the suppression level of the system to exogenous disturbance. Thus, the smaller γ will lead to better performance.

The main objective of this paper is to design an observer-based controller such that the states of the system and observer are positive and converge to the origin. The implementation of this issue will be seen in the next section.

3. H_∞ OBSERVER-BASED STABILIZATION

In this subsection, we discuss H_∞ observer-based stabilization of a time-delay switched positive system (1) with an observer-based controller. The observer-based controller design problem includes the positivity and stability of the augmented system (10), which is analyzed and dissected in the following two theorems and corollary. In Theorem 1, we discuss the necessary condition to check the feasibility of the problem. Also, in Theorem 2, some sufficient conditions and the corresponding synthesis for an existing feasible solution will be presented. The capabilities of the observer-based control approach proposed in this paper will be compared with the output feedback method in another paper through corollary 1.

Theorem 1. Augmented system (10) is positive and stable if for any $m \in M$ with respect to Metzler G_m, $L_m \succcurlyeq 0$ and $K_m \preccurlyeq 0$, the following inequalities are met:

$$[\underline{A}_m + \overline{B}_m K_m]_{ij} \geq 0, \qquad 1 \leq i \neq j \leq n \tag{15}$$

$$[G_m - \underline{B}_m K_m]_{ij} \geq 0, \qquad 1 \leq i \neq j \leq n \tag{16}$$

$$\underline{A}_m - L_m \overline{C}_m + \overline{B}_m K_m - G_m \succcurlyeq 0 \tag{17}$$

$$trace\left(\overline{A}_m + G_m + \left(\underline{B}_m - \overline{B}_m\right) K_m\right) < 0 \tag{18}$$

Proof: See proof in Appendix I.

Theorem 2. There exists an observer-based controller (5)-(6) for the system (1) that provides H_∞ stabilization and positivity of system (10), if there exist scalar $\gamma > 0, \varepsilon > 0$, matrices $P_m = diag\left[P_{1,m}, P_{2,m}\right] > 0$, $Q_m = diag\left[Q_{1,m}, Q_{2,m}\right] > 0$, a Metzler matrix G_m, $A_{1,m} \succcurlyeq 0$, $L_m \succcurlyeq 0$ and $K_m \preccurlyeq 0$ such that:

$$\begin{bmatrix} \Gamma_m & \overline{A}_{h,m} & P_m \mathbb{B}_m + \mathbb{C}_m^T \mathbb{K}_m^T & \varepsilon \mathbb{B}_m & P_m A_{w,m} \\ * & \tilde{Q}_m & 0 & 0 & 0 \\ * & * & -I & 0 & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & -\gamma^2 I \end{bmatrix} < 0$$
$$\forall m \in M \tag{19}$$

and inequalities (15), (16) and (17) from Theorem 1 must be held as follows:

$$[\underline{A}_m + \overline{B}_m K_m]_{ij} \geq 0 \qquad 1 \leq i \neq j \leq n$$
$$[G_m - \underline{B}_m K_m]_{ij} \geq 0 \qquad 1 \leq i \neq j \leq n$$
$$A_m - L_m \overline{C}_m + \overline{B}_m K_m - G_m \succcurlyeq 0$$

and the following two relations are given to ensure the stability of the system according to ADT,

$$Q_{m_2} \leq \mu Q_{m_1},\ P_{m_2} \leq \mu P_{m1},\ for\ m_1, m_2 \in M\ ,\ \mu \geq 1 \tag{20}$$

and the ADT satisfies

$$\tau_a > \frac{ln\mu}{\alpha} \quad ,\forall m \in M\ \alpha_m = \alpha > 0 \tag{21}$$

which

$$\Gamma_m = \mathbb{A}_m{}^T P_m + P_m \mathbb{A}_m - \varepsilon P_m \mathbb{B}_m \mathbb{B}_m{}^T - \varepsilon \mathbb{B}_m \mathbb{B}_m{}^T P_m + \overline{Q}_m \tag{22}$$

and

$$\mathbb{A}_m = \begin{bmatrix} \overline{A}_m & 0 \\ \underline{A}_m & 0 \end{bmatrix},\ \mathbb{B}_m = \begin{bmatrix} 0 & \underline{B}_m - \overline{B}_m & 0 & \underline{B}_m \\ 0 & \underline{B}_m - \overline{B}_m & -I & \underline{B}_m \end{bmatrix},$$
$$\tilde{Q}_m = \begin{bmatrix} -Q_{1,m}(1-d)e^{-\alpha\bar{h}} + A_{2,m}^T A_{2,m} & 0 \\ 0 & -Q_{2,m}(1-d)e^{-\alpha\bar{h}} \end{bmatrix}$$
$$\mathbb{K}_m = \begin{bmatrix} G_m & L_m & 0 & 0 \\ K_m & 0 & 0 & 0 \\ 0 & 0 & G_m & L_m \\ 0 & 0 & K_m & 0 \end{bmatrix},\ \mathbb{C}_m = \begin{bmatrix} 0 & I \\ 0 & 0 \\ I & -I \\ \underline{C}_m & 0 \end{bmatrix},$$
$$\overline{A}_{h,m} = \begin{bmatrix} P_{1,m} A_{1,m} + C_{2,m}^T A_{2,m} & 0 \\ 0 & P_{2,m} A_{1,m} \end{bmatrix} \tag{23}$$

Proof:
See proof in Appendix II.

Remark 3. The problem raised in this study is solved using Theorems 1 and 2. By Theorem 1, the necessary condition to LMI positive answers despite uncertainty is ensured, and by Theorem 2, the sufficient condition for the presence of a positive H_∞ observer-based controller and its exponential stability is guaranteed.

Remark 4. If the γ, α and ε design parameters are not selected correctly according to the dynamics of the switched system, the problem will not reach feasible solutions. There can be two resolutions to solve this problem: 1) Assigning proper values to the parameters of each mode $(\gamma_m, \alpha_m,\ \varepsilon_m)$. 2) Assigning the same proper values for all modes. In this work, we chose the second way.

Remark 5. It is necessary to declare that the presented design based on the Theorems 1 and 2 leads to stable and nonnegative $x\,(t)\,, \hat{x}(t)$. The augmented closed-loop system by combining (5) and (7) is:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = \begin{bmatrix} A_\sigma & B_\sigma K_\sigma \\ L_\sigma C_\sigma & G_\sigma \end{bmatrix} \begin{bmatrix} x\,(t) \\ \hat{x}(t) \end{bmatrix}$$
$$+ \begin{bmatrix} A_{1,\sigma} & 0 \\ 0 & A_{1,\sigma} \end{bmatrix} \begin{bmatrix} x\,(t-h(t)) \\ \hat{x}(t-h(t)) \end{bmatrix} + \begin{bmatrix} D_{1,\sigma} \\ 0 \end{bmatrix} \omega(t) \tag{24}$$

If this system is considered as a based model, no feasible solution exists for LMIs related to this problem and hence, (24) has been replaced by (10).

Remark 6. From Theorem 1 and Theorem 2, the controller gains K_m and the observer gains L_m and G_m can be solved by the following iterative algorithm in Flowchart 1. The conditions for entering this flowchart are that the system is observable and controllable, internally stable, and the parameter uncertainty does not exceed a certain boundary. Each of the above conditions can be checked in separate flowcharts.

In the above figure $\gamma > 0,\ \alpha > 0,\ \mu > 1$ and $\varepsilon > 0$ are design parameters, and "**No_subsystems**" is the number of subsystems or modes. By solving inequalities 15 to 19, in addition to obtaining the observer and controller parameters, P and Q matrices are also obtained for each mode, which are used to design the switching regime.

If we remove the interval uncertainty, time-varying delay, and disturbance from the system (1) and rewrite it simply as below:

$$\begin{cases} \dot{x}\left(t\right) = A_{\sigma(t)}x\left(t\right) + B_{\sigma(t)}u(t) \\ y = C_{\sigma(t)}x\left(t\right) \end{cases} \tag{25}$$

Then, dynamic observer-based controller for system (25) is exist as follows:

$$\begin{cases} \dot{\hat{x}}\left(t\right) = G_{\sigma(t)}x\left(t\right) + L_{\sigma(t)}y\left(t\right) \\ u\left(t\right) = K_{\sigma(t)}\hat{x}\left(t\right) \end{cases} \tag{26}$$

Corollary 1: For $\sigma\left(t\right) = m, m\epsilon\ M$, and for given Metzler matrices A_m, G_m and matrices $B_m \succcurlyeq 0, C_m \succcurlyeq 0$, and $L_m \succcurlyeq 0, K_m \preccurlyeq 0$ and $P_m = diag\left[P_{1,m}, P_{2,m}\right] > 0$, and for given scalars $\lambda > 0,\ \ \varepsilon > 0,\ \tau_a > 0$ and $\mu \geq 1$, such that:

$$\Xi_m = \begin{bmatrix} \mathbb{A}_m^T P_m + P_m\mathbb{A}_m - \varepsilon\mathbb{B}_m\mathbb{B}_m^T P_m - \varepsilon P_m\mathbb{B}_m\mathbb{B}_m^T & P_m\mathbb{B}_m + G_m^T K_m^T & \varepsilon\mathbb{B}_m \\ * & -I & 0 \\ * & 0 & -I \end{bmatrix} < 0 \tag{27}$$

$$\left[A_m + B_m K_m\right]_{ij} \geq 0 1 \leq i \neq j \leq n \tag{28}$$

$$\left[G_m - B_m K_m\right]_{ij} \geq 0 1 \leq i \neq j \leq n \tag{29}$$

$$A_m + B_m K_m - G_m - L_m C_m \succcurlyeq 0 \tag{30}$$

$$P_{m_2} \leq \mu P_{m1}, \quad for\ \sigma\left(t_k\right) = m_2,\ \sigma\left(t_k^-\right) = m_1 \in M \tag{31}$$

where,

$$\mathbb{A}_m = \begin{bmatrix} A_m & 0 \\ A_m & 0 \end{bmatrix}, \mathbb{B}_m = \begin{bmatrix} 0 & 0 & 0 & B_m \\ 0 & 0 & -I & B_m \end{bmatrix}$$

$$\mathbb{C}_m = \begin{bmatrix} 0 & I \\ 0 & 0 \\ I & -I \\ C_m & 0 \end{bmatrix}, \mathbb{K}_m = \begin{bmatrix} G_m & L_m & 0 & 0 \\ K_m & 0 & 0 & 0 \\ 0 & 0 & G_m & L_m \\ 0 & 0 & K_m & 0 \end{bmatrix} \tag{32}$$

then, system (25) is positively switched and exponentially stable with ADT $\tau_a > \frac{ln\mu}{\lambda}$.

Proof:

See proof in Appendix III.

4. Simulation Results

In the current section, we will discuss three examples: a numerical example (Case No.1), a practical example (Case No.2), and a comparative example (Case No.3). The cause of the presentation, for each of these examples is described in the related subsection. An Intel Core i5-520M 2.4 GHz (boost up to 2.93 GHz) processor was used to measure the computational time provided at the end of the examples.

4.1. Numerical example (Case No.1)

In this subsection, a numerical example is presented that supports the components of a positive compartmental switched model with all the factors stated in the model (1). This example challenges the approach presented in the study without considering the constraints of a practical system model and the limitations of choosing a system to compare the capabilities of the proposed method. An SPS is introduced in this example, which includes stable and unstable modes, time-varying delay, and external disturbance. P and Q matrices are interface matrices employed in system stability analysis. The related design parameters are shown in Table 1. Note that the values of the switching parameters are the same for both modes. Two subsystems are described with the following matrices:

$$\underline{A}_1 = \begin{bmatrix} -1.749 & 0.239 \\ 0.455 & -1.101 \end{bmatrix}, \overline{A}_1 = \begin{bmatrix} -1.613 & 0.361 \\ 0.545 & -0.699 \end{bmatrix},$$
$$\underline{A}_{1,1} = \begin{bmatrix} 0.045 & 0.005 \\ 0.075 & 0.055 \end{bmatrix}, \overline{A}_{1,1} = \begin{bmatrix} 0.055 & 0.015 \\ 0.085 & 0.065 \end{bmatrix}$$

$$\underline{B}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \overline{B}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \underline{C}_1 = \begin{bmatrix} 0.9 & 0 \end{bmatrix},$$
$$\overline{C}_1 = \begin{bmatrix} 1.1 & 0 \end{bmatrix}, D_{1,1} = \begin{bmatrix} 0.01 \\ 0 \end{bmatrix}, C_{2,1} = \begin{bmatrix} 0.1 & 0 \end{bmatrix},$$
$$A_{2,1} = \begin{bmatrix} 0.1 & 0 \end{bmatrix};$$

$$\underline{A}_2 = \begin{bmatrix} -1.749 & 0.239 \\ 0.455 & 0.200 \end{bmatrix}, \overline{A}_2 = \begin{bmatrix} -1.700 & 0.249 \\ 0.540 & 0.220 \end{bmatrix},$$
$$\underline{A}_{1,2} = \begin{bmatrix} 0.045 & 0.005 \\ 0.075 & 0.055 \end{bmatrix}, \overline{A}_{1,2} = \begin{bmatrix} 0.050 & 0.006 \\ 0.080 & 0.070 \end{bmatrix}$$

$$\underline{B}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \overline{B}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \underline{C}_2 = \begin{bmatrix} 0.8 & 0 \end{bmatrix},$$
$$\overline{C}_2 = \begin{bmatrix} 1.0 & 0 \end{bmatrix}, D_{1,2} = \begin{bmatrix} 0.01 \\ 0 \end{bmatrix}, C_{2,2} = \begin{bmatrix} 0.01 & 0 \end{bmatrix},$$
$$A_{2,2} = \begin{bmatrix} 0.01 & 0 \end{bmatrix};$$

And time-varying delay and disturbance respectfully are calculated as follows:

$$h\left(t\right) = \overline{h}/2(1 + \frac{sin(dt)}{\overline{h}}), \omega\left(t\right) = e^{-0.2t}.$$

Yalmip toolbox [43] was employed in MATLAB software to obtain simulation results in all examples. Solving the matrix inequalities in Theorems 2 gives feasible answers as follows

Table 1. Design parameters of the case no. 1

Parameters	Values
γ	0.5
d	0.6
α	0.6
ε	100
$\overline{h}$	0.1
μ	1.6

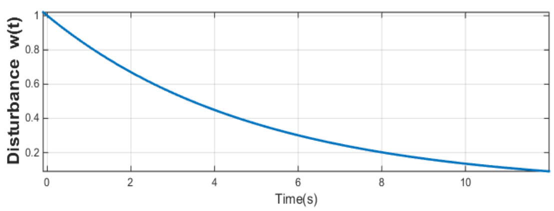

Fig. 3. Disturbance signal

$$K_1 = \begin{bmatrix} -1.4977 & -0.055371 \\ -0.13023 & -0.43027 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -17772 & -0.066049 \\ -0.10692 & -19.958 \end{bmatrix}$$

$$L_1 = \begin{bmatrix} 6.5454 \\ 0.10711 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 72023 \\ 0.096979 \end{bmatrix}$$

$$G_1 = \begin{bmatrix} -10.753 & 0.1318 \\ 0.14497 & -1.6129 \end{bmatrix}, \quad G_2 = \begin{bmatrix} -94832 & 0.1069 \\ 0.1542 & -39.569 \end{bmatrix}$$

$$P_{1,1} = \begin{bmatrix} 16.483 & -0.86641 \\ -0.86641 & 15.357 \end{bmatrix}, \quad P_{1,2} = \begin{bmatrix} 18.797 & -0.21121 \\ -0.21121 & 14.961 \end{bmatrix}$$

$$P_{2,1} = \begin{bmatrix} 1.3349e+05 & -0.21169 \\ -0.21169 & 1.4e+05 \end{bmatrix}, \quad P_{2,2} = \begin{bmatrix} 1.7214e+05 & -0.22283 \\ -0.22283 & 1.4e+05 \end{bmatrix}$$

$$Q_{1,1} = \begin{bmatrix} 43.218 & -10.1 \\ -10.1 & 8.7193 \end{bmatrix}, \quad Q_{1,2} = \begin{bmatrix} 47.587 & -5.9433 \\ -5.9433 & 6.214 \end{bmatrix}$$

$$Q_{2,1} = \begin{bmatrix} 1.407e+09 & -79086 \\ -79086 & 14416 \end{bmatrix}, \quad Q_{2,2} = \begin{bmatrix} 2.7257e+09 & -78249 \\ -78249 & 14416 \end{bmatrix}$$

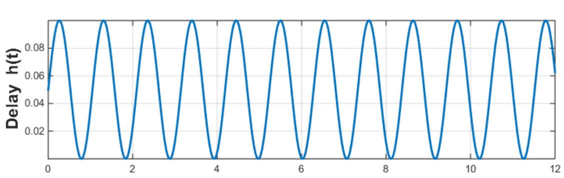

Fig. 4. Time-varying delay

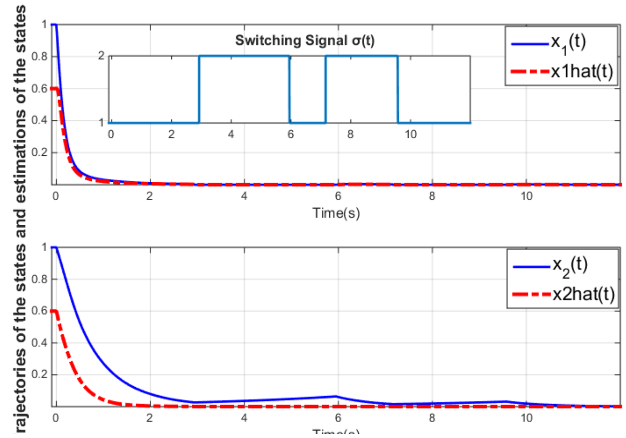

Fig. 5. State trajectories and estimation

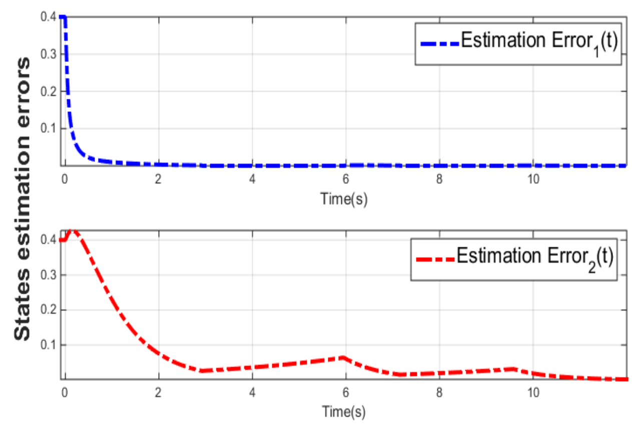

Fig. 6. Error signal for 12 seconds running time

Fig. 3 represents the treatment of exogenous disturbance. Fig. 4 shows the behavior of the time-varying delay function, which is sinusoidal, and this behavior is very close to the delay variations in a practical system.

Then, we present simulation results for ADT switching $\tau_a \geq 0.78$ second for two subsystems. The simulation results are shown in Fig. 5 where the initial conditions are $x(0) = [1\ 1]^T$, $\hat{x}(0) = [0.6\ 0.6]^T$. The switching signal is designed by keeping the average dwell time in Fig. 5 for 12 seconds. As can be seen in Fig 5, the controllers and observers are well designed for both stable and unstable states, and both state variables have converged to zero within about 10 seconds of execution. During this period, the estimates of state variables have also maintained their tracing trend. In Fig. 6, the trajectories of the error signal between the state variables ($x_1(t)$, $x_2(t)$) and their estimations are drawn, which converge to zero. Parametric uncertainty has also been applied during execution in the form of changing parameter values in the desired range continuously. The time elapsed for the calculations of case no.1 was approximately 30 seconds.

4.2. Practical example (Case No.2)

In this example, a compartmental model is considered a system with positive switching characteristics, disturbance, time-varying delay, and parametric uncertainty for analyzing the dynamics of thyroid hormone metabolism [43]. As shown in Fig. 9, the Mammillary model consists of three compartments, each representing a state variable. State variables are the quantities of hormones in each compartment. The two composite tissue pools (fast and slow) represent two classes of organ pools of thyroid

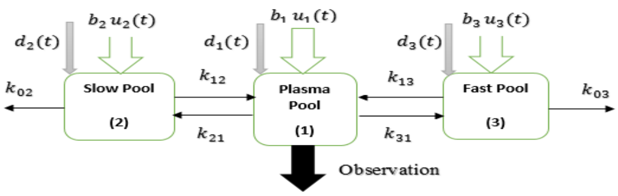

Fig. 7. General Mammillary compartmental model

hormone such as T3 and T4 [44] (Peripheral tissues such as the liver, kidneys, brain, and pituitary gland can also convert T4 to T3). The plasma pool is the central compartment. Each state variable denotes the quantity of the hormone in the pool. Each tissue group metabolizes or disposes of thyroid hormone. By measuring the amount of thyroid hormone in the blood plasma with the aid of an observer, the amount of this hormone in other tissues can be estimated. The above compartmental model can be described as a switching model. It means that we have different coefficients for the same structures and can be switched between them. In [45], a compartmental switching system for controlling the amount of anesthetic is discussed. Factors such as taking certain medications or psychosomatic illnesses such as mood disorders can cause these different subsystems. A compartmental system can be a time delay system. This is the subject of [46] and has been discussed in [47]. The effect of external disturbances on the compartmental system has been investigated in [48]. These disturbances in the current system can be caused by various factors such as hereditary and environmental factors such as nutrition, stress, smoking, iodine deficiency, and other underlying diseases.

The mathematical model of the system is similar to the system (1) and is as follows:

$$\begin{aligned}
A_1 &= \begin{bmatrix} -k_{21}-k_{31} & k_{12} & k_{13} \\ k_{21} & -k_{02}-k_{12} & 0 \\ k_{31} & 0 & k_{03}-k_{13} \end{bmatrix}, \\
A_{1,1} &= \begin{bmatrix} 0 & 0 & 0.02 \\ 0.01 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
B_1 &= \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}, \\
C_1 &= \begin{bmatrix} k_c & 0 & 0 \end{bmatrix} \\
D_{1,w} &= \begin{bmatrix} 0.02 \\ 0.01 \\ 0.01 \end{bmatrix}
\end{aligned} \tag{33}$$

$$\begin{aligned}
A_2 &= \begin{bmatrix} -k_{21}-k_{31} & k_{12} & k_{13} \\ k_{21} & -k_{02}-k_{12} & 0 \\ k_{31} & 0 & -k_{03}-k_{13} \end{bmatrix}, \\
A_{1,2} &= \begin{bmatrix} 0 & 0 & 0 \\ 0.02 & 0 & 0.03 \\ 0 & 0 & 0 \end{bmatrix}, \\
B_2 &= \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}, \\
C_2 &= \begin{bmatrix} k_c & 0 & 0 \end{bmatrix} \\
D_{2,w} &= \begin{bmatrix} 0.03 \\ 0.01 \\ 0.04 \end{bmatrix}
\end{aligned} \tag{34}$$

The k_{ij} parameters of the model are estimated to be obtained with the help of the observer or reliable references. The values of these parameters represent interval uncertainty in the system matrices. Table 2 lists these values for both modes.

Table 2. Uncertain parameters of the model

Second mode	First mode	Parameters
2.6080±0.0802	1.8090 ± 0.0420	k_{21}
1.6607±0.0505	1.5550 ± 0.0630	k_{31}
2.2011±0.0103	2.0650 ± 0.0610	k_{12}
1.4512±0.0308	1.2550 ± 0.0450	k_{13}
0.8051±0.0310	0.7050 ± 0.1010	k_{02}
1.2501±0.1399	1.2310 ± 0.1400	k_{03}
0.8±0.1	1 ± 0.1	k_c

Table 3. Design parameters for case no.2

Parameters	Values
γ	0.5
d	0.9
α	0.7
ε	100
$\bar{h}$	1

By applying Theorem 2 for the current example, using the Yalmip toolbox, the following gains for the controller and observer have been obtained:

$$K_1 = \begin{bmatrix} -14.73 & -0.67 & -0.41 \\ -0.85 & -8.57 & 0.00 \\ -0.67 & 0.00 & -4.49 \end{bmatrix}, \quad L_1 = \begin{bmatrix} 50.05 \\ 0.34 \\ 0.29 \end{bmatrix},$$
$$G_1 = \begin{bmatrix} -73.93 & 0.92 & 0.55 \\ 0.59 & -13.40 & 0.00 \\ 0.52 & 0.00 & -8.73 \end{bmatrix}$$

$$K_2 = \begin{bmatrix} -15.08 & -1.86 & -1.48 \\ -5.26 & -6.17 & 0.00 \\ -4.33 & 0.00 & -5.08 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 38.43 \\ 0.44 \\ 0.33 \end{bmatrix},$$
$$G_2 = \begin{bmatrix} -40.86 & 1.51 & 0.95 \\ 0.41 & -6.16 & 0.00 \\ 0.33 & 0.00 & -4.21 \end{bmatrix}$$

It should be noted that all time units in this example are in minutes. If ADT is calculated in minutes and a plasma compartment is used for measurement, effective thyroid drugs (controller) can be employed according to the defined target. Disturbance and Time-varying delay (Fig. 8 and Fig. 9) defined respectfully as: $h(t) = \bar{h}/2(1 + \frac{sin(dt)}{h})$. The wgn(), butter(), and filter() functions in MATLAB are used to generate disturbance $\omega(t)$. As can be seen in the figure, the disturbance implementation is such that it has all the nonnegative values that a positive system must have for this example. Figs 10 and 11 shows parameter uncertainty in matrices A and C, respectively. The switching signal is given in Fig. 12. The initial values for states and their estimations are, respectfully: $x(t) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$, $\hat{x}(0) = \begin{bmatrix} 0.7 & 0.7 & 0.7 \end{bmatrix}^T$ and the other related parameters has been shown in Table 3.

The trend of Fig. 12 curves can be interpreted as follows, for one cycle, thyroid hormone enters the plasma pool and then enters the slow and fast pools. The amount of hormone in each of them is metabolized and finally excreted. The controller and observer that measures the plasma pool play an important role in this process. The results of reference [44] are the same in general. The time elapsed for the calculations of this example was about 0.5 second.

4.3. Comparative example (Case No.3)

In this subsection, we wanted to compare the capabilities of L_1-gain (with relatively similar models and from other references) and L_2-gain (described in this paper) methods in observer-based controller design, such as the number of iterations

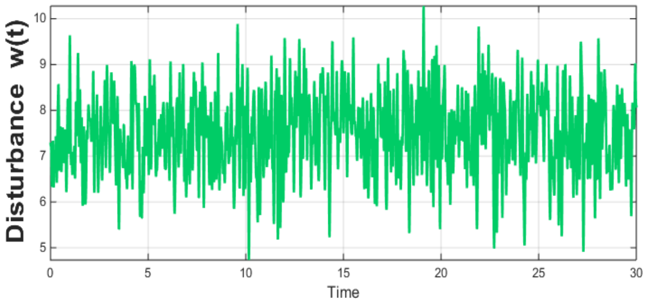

Fig. 8. Disturbance signal for case no. 2

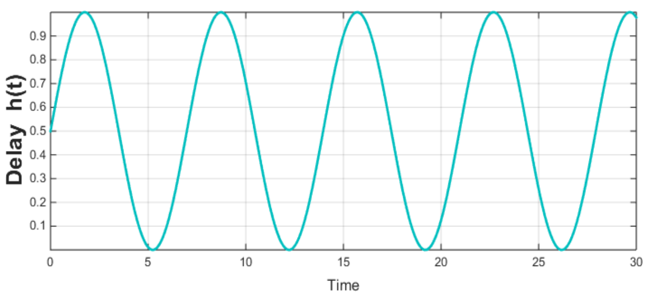

Fig. 9. Time-varying delay for case no. 2

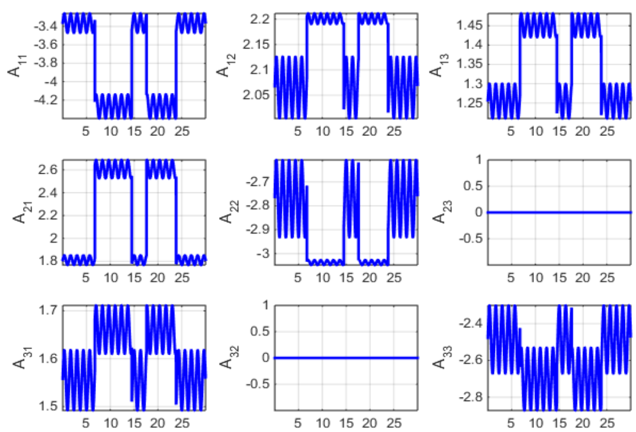

Fig. 10. Parameter uncertainties of A matrix elements

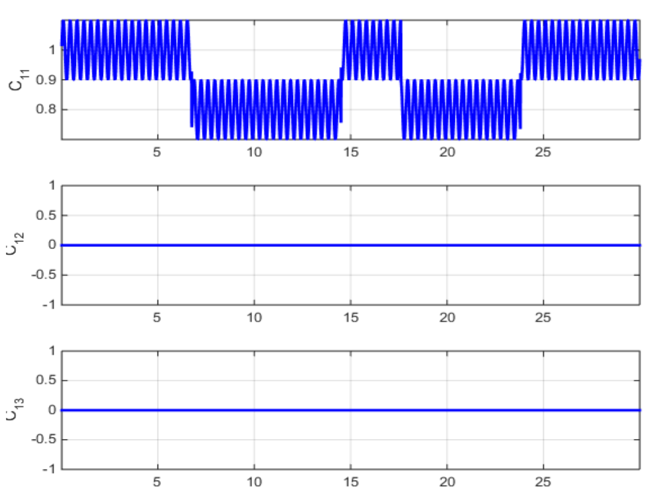

Fig. 11. Parameter uncertainties of A matrix elements

Table 4. Comparison of MSE values by two methods ($\overline{T} = 5\ sec$)

	Liu [49]	Corollary 1
MSE of x_1(t)	0.081	0.008
MSE of x_2(t)	0.048	0.034
MSE of x_3(t)	0.058	0.032

required to solve the problem, error, and performance. But, for the proposed dynamics, none of the methods introduced in the literature [36, 38, 41] became feasible hence, there was no possibility for comparison. The reference method [48], despite the difference in model components such as delay and parametric uncertainty with the model of this article and also the use of the Linear Programming (LP) method instead of L_2-gain and Lyapunov function, can be compared with the approach of this paper. It is because the existence of commonalities, such as positivity, linearity, switching, ADT design, use of output feedback in controller design, and most importantly, feasible solutions. In other cases, without feasible answers, the accuracy of the controller and observer performance cannot be trusted. Corollary1, was created to make this comparison. Theorem (3.1) from [48] was implemented, with the control law described below (p is the mode number), and the results were compared.

$$\begin{aligned} u(t) &= \overline{G}_p y(t) \\ &= k_p v_p y(t) \\ &= \frac{1}{v_p C_p \lambda_p} z_p v_p C_p x(t) \quad k_p \epsilon R^{m \times l}, \lambda_p \epsilon R^n, z_p \epsilon R^m \end{aligned} \tag{35}$$

v_p is a fixed parameter. The following system matrices were simulated for both methods.

$$A_1 = \begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.4 \end{bmatrix}$$
$$C_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -4 & 2 & 1 \\ 2 & -3 & 0 \\ 1 & 0 & -2 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}$$
$$C_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

The simulation results for the first mode are as follows ($\overline{G}_m$ is control gain matrix in [48]):

$$K_1 = \begin{bmatrix} -58.65 & -1.84 & -1.14 \\ -2.18 & -25.44 & 0.00 \\ -0.92 & 0.00 & -20.94 \end{bmatrix}, \quad L_1 = \begin{bmatrix} 178 \\ 0.38 \\ 0.33 \end{bmatrix},$$
$$G_1 = \begin{bmatrix} -219.59 & 0.94 & 0.56 \\ 0.60 & -21.40 & 0.00 \\ 0.52 & 0.00 & -24.91 \end{bmatrix}$$

$$\overline{G}_1 = \begin{bmatrix} -1.62 \\ -6.35 \\ -2.24 \end{bmatrix}, \ z_1 = \begin{bmatrix} -51.42 \\ -23.60 \\ -11.84 \end{bmatrix}, \ \lambda_1 = \begin{bmatrix} 9.45 \\ 5.57 \\ 9.43 \end{bmatrix}$$

and for the second mode are:

$$K_2 = \begin{bmatrix} -79.82 & -4.79 & -3.53 \\ -2.25 & -21.00 & 0.00 \\ -2.08 & 0.00 & -23.48 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 319.12 \\ 0.71 \\ 0.44 \end{bmatrix},$$
$$G_2 = \begin{bmatrix} -283.59 & 1.11 & 0.69 \\ 0.85 & -20.77 & 0.00 \\ 0.55 & 0.00 & -19.08 \end{bmatrix}$$

$$\overline{G}_2 = \begin{bmatrix} 2.27 \\ -8.10 \\ -5.82 \end{bmatrix}, \ z_2 = \begin{bmatrix} -41.57 \\ -68.54 \\ -31.73 \end{bmatrix}, \ \lambda_2 = \begin{bmatrix} 11.78 \\ 6.15 \\ 10.79 \end{bmatrix}$$

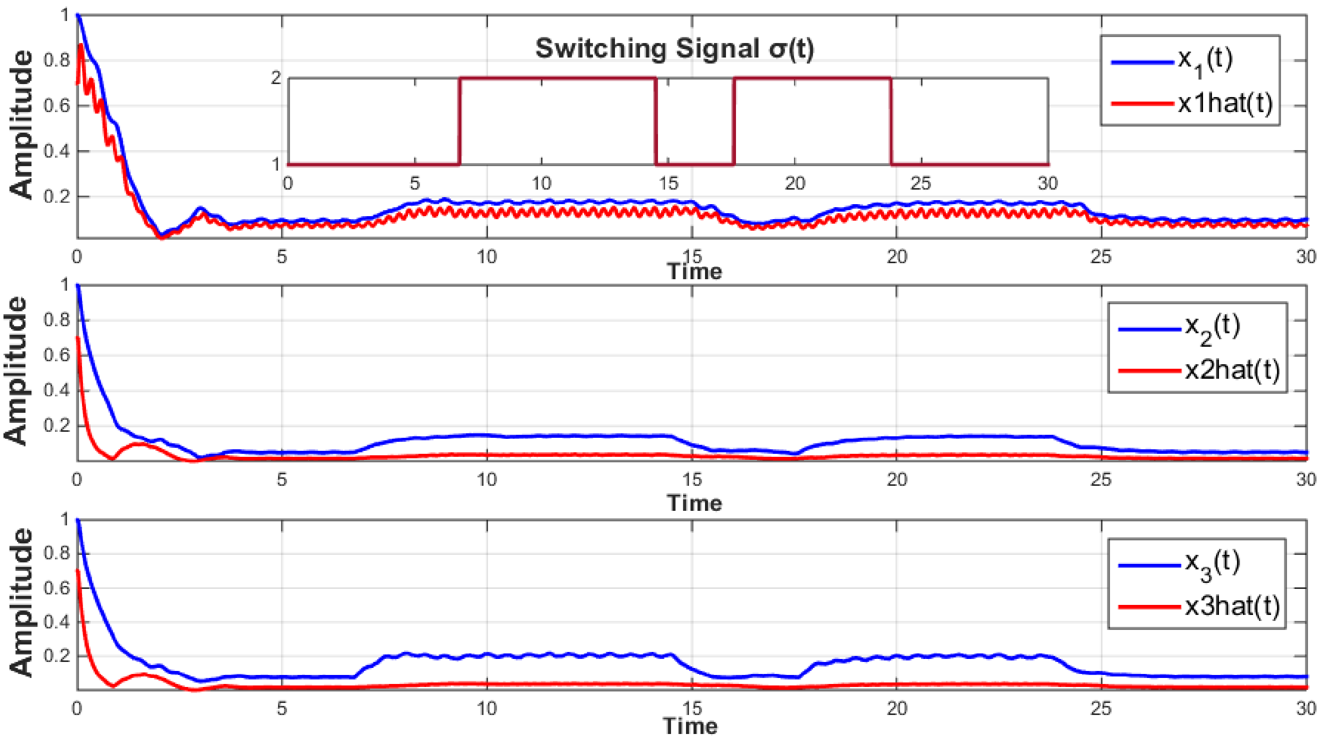

Fig. 12. States and their estimations

The switching signal for this example is presented in Fig. 13. The convergence of the modes of each method is also shown in the same figure. As clearly seen in the figure, the rate of convergence in the approach of this paper is more suitable than the procedure of [49]. In Table 4, the mean square error (MSE) values of both methods are presented for all three state variables. The method of calculating MSE for each mode is according to Equation 34. $e(t)$ is the vector of values of the errors between the current time and the predicted values.

$$MSE = \sqrt{\int_0^T e(t)^T e(t)dt} \tag{36}$$

Employing an observer-based controller for an SPS using the output feedback method is not very common, as mentioned earlier. The few cases that exist in the literature have no feasible solution and cannot be presented in a comparative example. Liu et al. [49] was the only case that was observed to have feasible solution by the output feedback controller and LP approach. The execution time of both algorithms was approximately 0.1 second.

5. Conclusions

In this paper, an effective method for designing a positive augmented closed-loop switched system consisting of an observer and a controller based on output feedback and state estimation was presented. In addition, minimizing the effects of destructive factors that generally exist in practical systems, such as parametric uncertainty, time-varying delay, disturbance, and unstable modes, has been one of the design goals. This methodology, especially for solving augmented systems, produces a large number of interface parameters in the form of positive vectors, which can be converted to BMI terms in the solution process. BMI terms either should be linearized with algebraic methods, which is not always possible, or solved with special-purpose solvers that do not have trustworthy answers, which in both cases does not provide a reliable design for the observer and the controller. In the proposed method that

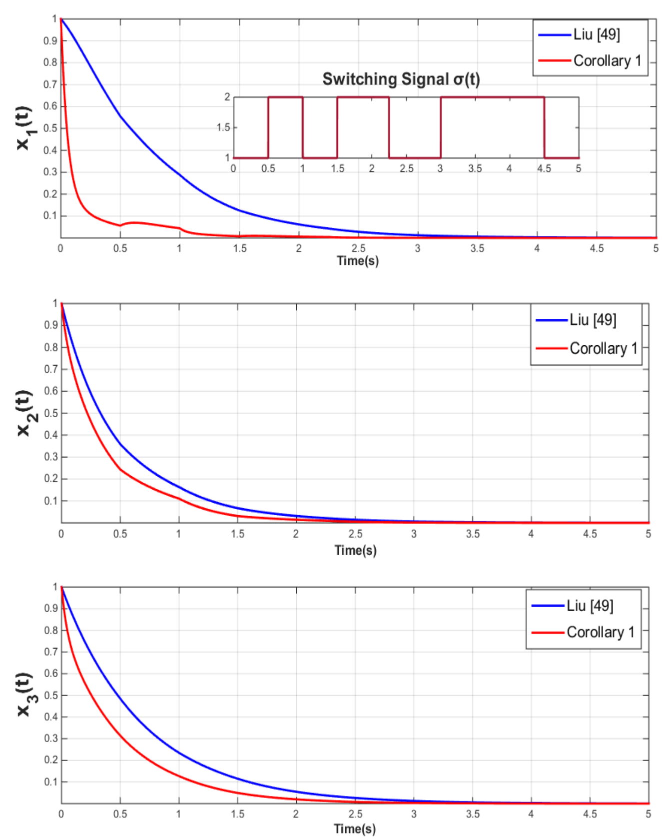

Fig. 13. Three states convergence with both methods

leads to H_∞ stabilization, the solution method consists of a set of LMI terms.

For further validation of the proposed approach, three examples were illustrated. In the first case, a numerical example was evaluated. In this case, in addition to reducing the effect of parametric uncertainties and disturbance and delay, it was noticed that the robust controller and the ADT switching regime did not allow the divergence of the trajectories of the state variables when the unstable mode was activated. Also, the gains of the controller and observer were obtained according to the design assumptions, and the diagonals of the matrices P and Q were positive, which are used to design the switching regime. This example best demonstrated the achievement of our robust observer-based control system. In the second case, a positive compartmental switched system was presented, in which the variables of thyroid hormone metabolism states were considered in three organs of the body with high and low metabolism and blood plasma. This system had high disturbance, parametric uncertainty, and delay in state variables. By measuring the hormone in the blood, its status in other organs can be estimated and controlled. In this case, all the state variables converged to the origin, but due to the bias in the disturbance, a small amount of steady-state error remained. By the presented method, after about 27 minutes, the amount of hormone in the tissue reaches the desired amount.

In order to evaluate the performance of the proposed observer-based controller, there was a condition for a similar system, a system with a similar model should be used in comparison (output feedback). Among the existing papers in this field, [48] was selected because of feasible answers. On the other hand, Corollary 1 was generated in Section 3 for comparison. As can be seen in the figures of the third example Fig. 13, the convergence rate of our proposed method was much better than that of the reference method [48]. The MSE of the system output $(y = x_1(t))$ for the method of this paper is 0.008 and for the compared approach is 0.081. Also, this value for two other state variables obtained by estimation with values of 0.034 and 0.0032 is more acceptable than the method [48] with values of 0.048 and 0.0058, respectively (see Table 4). In this example, the advantage of employing dynamic output feedback over static output feedback was also shown.

6. Appendixes

Appendix I.

If there exists such an observer-based controller from (10), for stability of the closed-loop system we should have $\vartheta(A_{x,m}) < 0$ and for positiveness $A_{x,m}$ should be Metzler. Thus for any $m \in M$, $A_m \in [\underline{A_m}, \overline{A_m}]$, $B_m \in [\underline{B_m}, \overline{B_m}]$ and $C_m \in [\underline{C_m}, \overline{C_m}]$, with respect to $K_m \preccurlyeq 0$, we have:

$$\begin{bmatrix} \underline{A}_m + \overline{B}_m K_m & -\underline{B}_m K_m \\ \underline{A}_m + \overline{B}_m K_m - L_m \overline{C}_m - G_m & G_m - \underline{B}_m K_m \end{bmatrix} \preccurlyeq \begin{bmatrix} A_m + B_m K_m & -B_m K_m \\ A_m + B_m K_m - L_m C_m - G_m & G_m - B_m K_m \end{bmatrix} \preccurlyeq \begin{bmatrix} \overline{A}_m + \underline{B}_m K_m & -\overline{B}_m K_m \\ \overline{A}_m + \underline{B}_m K_m - L_m \underline{C}_m - G_m & G_m - \overline{B}_m K_m \end{bmatrix} \tag{A1}$$

Then, it can be interfered:

$$\vartheta\left(\begin{bmatrix} \overline{A}_m + \underline{B}_m K_m & -\overline{B}_m K_m \\ \overline{A}_m + \underline{B}_m K_m - L_m \underline{C}_m - G_m & G_m - \overline{B}_m K_m \end{bmatrix}\right) < 0 \tag{A2}$$

and $\begin{bmatrix} \underline{A}_m + \overline{B}_m K_m & -\underline{B}_m K_m \\ \underline{A}_m + \overline{B}_m K_m - L_m \overline{C}_m - G_m & G_m - \underline{B}_m K_m \end{bmatrix}$ should be Metzler which guarantees (15), (16) and (17). Eq. (A2) indicates that all the eigenvalues of $A_{x,m}$ are negative. From (A2) in conformity with Lemma 2, the same condition is held for $A_{x,m}$ and we have:

$$trace\left(\begin{bmatrix} \overline{A}_m + \underline{B}_m K_m & -\overline{B}_m K_m \\ \overline{A}_m + \underline{B}_m K_m - L_m \underline{C}_m - G_m & G_m - \overline{B}_m K_m \end{bmatrix}\right) < 0 \tag{A3}$$

Inequality (A3) is equivalent to (18) and the proof is complete.

Appendix II.

The Lyapunov-Krasovskii function can be constructed as follow:

$$\begin{aligned} V_\sigma(t) &= V(t) \\ &= \tilde{x}^T(t) P_{\sigma(t)} \tilde{x}(t) + \int_{t-h(t)}^{t} e^{-\alpha(t-s)} \tilde{x}(t) Q_{\sigma(t)} \tilde{x}(t) ds \end{aligned} \tag{A4}$$

For $\sigma(t) = m$, by substituting $P_m = diag[P_{1,m}, P_{2,m}] > 0$ and $Q_m = diag[Q_{1,m}, Q_{2,m}] > 0$, we have:

$$\begin{aligned} V(t) &= x^T P_{1,m} x + e^T P_{2,m} e \\ &+ \int_{t-h(t)}^{t} e^{-\alpha(t-s)} \Big[x^T(s) Q_{1,m} x(s) \\ &+ e^T(s) Q_{2,m} e(s)\Big] \mathrm{d}s \end{aligned} \tag{A5}$$

By derivation of (A5), we can get:

$$\begin{aligned} \dot{V}(t) &= x^T(t) P_{1,m} \dot{x}(t) + \dot{x}^T(t) P_{1,m} x(t) \\ &+ e^T(t) P_{2,m} \dot{e}(t) + \dot{e}^T(t) P_{1,m} e(t) + x^T(t) Q_{1,m} x(t) \\ &+ e^T(t) Q_{2,m} e(t) - \alpha\left(V - x^T(t) P_{1,m} x(t) - e^T(t) P_{2,m} e(t)\right) \\ &- \left(1 - \dot{h}\right) e^{-\alpha h(t)} [x^T(t - h(t) Q_{1,m} x(t - h(t)) \\ &+ e^T(t - h(t)) Q_{2,m} e(t - h(t))] \end{aligned} \tag{A6}$$

We can conclude:

$$\dot{V}(t) + \alpha V(t) \leq \eta^T(t) \varphi_m \eta(t) \tag{A7}$$

Which guarantees the stability of the system $(\omega = 0)$, where $\eta(t)$ is a matrix consisting of several vectors:

$$\eta^T(t) = \begin{bmatrix} x^T(t) & e^T(t) & x^T(t - h(t)) & e^T(t - h(t)) & \omega^T \end{bmatrix} \tag{A8}$$

$$\varphi_m = \begin{bmatrix} \pi_{11} & \pi_{12} & P_{1,m} A_{1,m} & 0 & P_{1,m} D_{1,m} \\ \pi_{21} & \pi_{22} & 0 & P_{2,m} A_{1,m} & P_{2,m} D_{1,m} \\ A_{1,m}^T P_{1,m} & 0 & -(1-d) e^{-\alpha \overline{h}} Q_{1,m} & 0 & 0 \\ 0 & A_{1,m}^T P_{2,m} & 0 & -(1-d) e^{-\alpha \overline{h}} Q_{2,m} & 0 \\ D_{1,m}^T P_{1,m} & D_{1,m}^T P_{2,m} & 0 & 0 & 0 \end{bmatrix} \leq 0 \tag{A9}$$

$$\pi_{11} = P_{1,m}\left(A_m + B_m K_m\right) + Q_{1,m} + \left(A_m + B_m K_m\right)^T P_{1,m} + \alpha P_{1,m}$$
$$\pi_{12} = \left(A_m + B_m K_m - L_m C_m - G_m\right)^T P_{2,m} - P_{1,m} B_m K_m$$
$$\pi_{21} = P_{2,m}\left(A_m + B_m K_m - L_m C_m - G_m\right) - K_m^T B_m^T P_{1,m}$$
$$\pi_{22} = P_{2,m}\left(G_m - B_m K_m\right) + Q_{2,m} + \left(G_m - B_m K_m\right)^T P_{2,m} + \alpha P_{2,m}$$

For disturbance attenuation by means of L_2-gain stability relation $\frac{\|z(t)\|_{L_2}}{\|\omega(t)\|_{L_2}} \le \gamma$, it can be stated that:

$$\frac{\sqrt{\int_0^\infty z^T(t) z(t)\, dt}}{\sqrt{\int_0^\infty \omega(t)^T \omega(t)}} \le \gamma \tag{A10}$$

L_2-gain stability criterion in (A10) can rewritten in following form:

$$J = \int_0^\infty z^T(t) z(t) - \gamma^2 \omega^T(t) \omega(t)\, dt \le 0 \tag{A11}$$

According to Definition 4, we can conclude

$$J \le \int_0^\infty z^T(t) z(t) - \gamma^2 \omega^T(t) \omega(t)\, dt + V(x(\infty)) - V(x(0)) \le 0 \tag{A12}$$

Which yields:

$$J \le \int_0^\infty (z^T(t) z(t) - \gamma^2 \omega^T(t) \omega(t) + \dot{V}(t)) dt \le 0 \tag{A13}$$

A sufficient condition to guarantee the above inequality is:

$$z^T(t) z(t) - \gamma^2 \omega^T(t) \omega(t) + \dot{V}(t) \le 0 \tag{A14}$$

With respect to (A7), (A15) is rewritten as:

$$\begin{aligned} & z^T(t) z(t) - \gamma^2 \omega^T(t) \omega(t) + \dot{V}(t) \\ & \le z^T(t) z(t) - \gamma^2 \omega^T(t) \omega(t) + \dot{V}(t) + \alpha V(t) \\ & \le z^T(t) z(t) - \gamma^2 \omega^T(t) \omega(t) + \eta^T(t) \varphi_m \eta(t) \end{aligned} \tag{A15}$$

By considering the system dynamic (1), $z(t) = C_{2,\sigma(t)} x(t) + A_{2,\sigma(t)} x(t - h(t))$, we have:

$$\begin{aligned} & x^T(t) C_{2,m}^T C_{2,m} x(t) + x^T C_{2,m}^T A_{2,m} x(t - h(t)) \\ & + x^T(t - h(t)) A_{2,m}^T C_{2,m} x(t) + x^T(t - h) A_{2,m}^T A_{2,m} x(t) \\ & \quad - \gamma^2 \omega^T \omega(t) + \eta^T(t) \varphi_m \eta(t) \le \eta^T(t) \psi_m \eta(t) \end{aligned} \tag{A16}$$

So, we should have $\psi_m \le 0$ in which:

$$\psi_m = \begin{bmatrix} \chi_{11} & \chi_{12} & P_{1,m} A_{1,m} + C_2^T A_{2,m} & 0 & P_{1,m} D_{1,m} \\ \chi_{21} & \chi_{22} & 0 & P_{2,m} A_{1,m} & P_{2,m} D_{1,m} \\ A_{1,m}^T P_{1,m} + A_{2,m}^T C_2 & 0 & -(1-d) e^{-\alpha \bar{h}} Q_{1,m} + A_{2,m}^T A_{2,m} & 0 & 0 \\ 0 & A_{1,m}^T P_{2,m} & 0 & -(1-d) e^{-\alpha \bar{h}} Q_{2,m} & 0 \\ D_{1,m}^T P_{1,m} & D_{1,m}^T P_{2,m} & 0 & 0 & -\gamma^2 I \end{bmatrix} \le 0 \tag{A17}$$

$$\chi_{11} = P_{1,m}\left(A_m + B_m K_m\right) + Q_{1,m} + \left(A_m + B_m K_m\right)^T P_{1,m} + C_{2,m}^T C_{2,m} + \alpha P_{1,m}$$
$$\chi_{12} = \left(A_m + B_m K_m - L_m C_m - G_m\right)^T P_{2,m} - P_{1,m} B_m K_m + \alpha P_{2,m}$$
$$\chi_{21} = P_{2,m}\left(A_m + B_m K_m - L_m C_m - G_m\right) - K_m^T B_m^T P_{1,m}$$
$$\chi_{22} = P_{2,m}\left(G_m - B_m K_m\right) + Q_{2,m} + \left(G_m - B_m K_m\right)^T P_{2,m}$$

By substituting (23) into (A17) with respect to upper bound of the uncertain parameters, matrix ψ_m can be rewritten as follows,

$$\psi_m = \begin{bmatrix} \left(\mathbb{A}_m + \mathbb{B}_m \mathbb{K}_m \mathbb{C}_m\right)^T P_m + P_m \left(\mathbb{A}_m + \mathbb{B}_m \mathbb{K}_m \mathbb{C}_m\right) + \overline{Q}_m & \overline{A}_{h,m} & P_m A_{w,m} \\ * & \tilde{Q}_m & 0 \\ * & * & -\gamma^2 I \end{bmatrix} \le 0 \tag{A18}$$

where

$$\overline{Q}_m = \begin{bmatrix} Q_{1,m} + C_{2,m}^T C_{2,m} + \alpha P_{1,m} & 0 \\ 0 & Q_{2,m} + \alpha P_{2,m} \end{bmatrix} \tag{A19}$$

$$\tilde{Q}_m = \begin{bmatrix} -Q_{1,m}(1-d) e^{-\alpha \bar{h}} + A_{2,m}^T A_{2,m} & 0 \\ 0 & -Q_{2,m}(1-d) e^{-\alpha \bar{h}} \end{bmatrix} \tag{A20}$$

$$\begin{aligned} \Sigma_m &= \mathbb{A}_m + \mathbb{B}_m \mathbb{K}_m \mathbb{C}_m \\ &= \begin{bmatrix} \overline{A}_m + \underline{B}_m K_m & -\overline{B}_m K_m \\ \overline{A}_m + \underline{B}_m K_\sigma - L_m \underline{C}_m - G_m & G_m - \overline{B}_m K_m \end{bmatrix} \end{aligned} \tag{A21}$$

By the means of Schur compliment [28], (A18) can be given in the following form:

$$\begin{bmatrix} {\Sigma_m}^T P_m + P_m \Sigma_m + \overline{Q}_m & \overline{A}_{h,m} \\ * & \tilde{Q}_m \end{bmatrix} + \begin{bmatrix} P_m A_{w,m} \\ 0 \end{bmatrix} \left(-\gamma^{-2} I\right) \begin{bmatrix} A_{w,m}^T P_m & 0 \end{bmatrix} \le 0 \tag{A22}$$

Again, using Schur compliment yields:

$$\begin{aligned} & {\Sigma_m}^T P_m + P_m \Sigma_m + \overline{Q}_m - \gamma^{-2} P_m A_{w,m} A_{w,m}^T P_m \\ & \quad - \overline{A}_{h,m} \tilde{Q}_m^T \overline{A}_{h,m}^T \le 0 \end{aligned} \tag{A23}$$

The following inequality can be considered as a sufficient condition for (A23):

$$\Sigma_m{}^T P_m + P_m \Sigma_m + \overline{Q}_m - \gamma^{-2} P_m A_{w,m} A_{w,m}^T P_m - \overline{A}_{h,m} \tilde{Q}_m^T \overline{A}_{h,m}^T + \mathbb{C}_m^T \mathbb{K}_m^T \mathbb{K}_m \mathbb{C}_m \leq 0 \quad \text{(A24)}$$

After replacement $\Sigma_m = \mathbb{A}_m + \mathbb{B}_m \mathbb{K}_m \mathbb{C}_m$, yields:

$$\mathbb{A}_m^T P_m + P_m \mathbb{A}_m + P_m \mathbb{B}_m \mathbb{K}_m \mathbb{C}_m + \mathbb{C}_m^T \mathbb{K}_m^T \mathbb{B}_m^T P_m + \mathbb{C}_m^T \mathbb{K}_m^T \mathbb{K}_m \mathbb{C}_m + \overline{Q}_m - \gamma^{-2} P_m A_{w,m} A_{w,m}^T P_m - \overline{A}_{h,m} \tilde{Q}_m^T \overline{A}_{h,m}^T \leq 0 \quad \text{(A25)}$$

which can be rewritten in this form:

$$\mathbb{A}_m^T P_m + P_m \mathbb{A}_m + P_m \mathbb{B}_m \mathbb{K}_m \mathbb{C}_m + \left(P_m \mathbb{B}_m + \mathbb{C}_m^T \mathbb{K}_m^T\right)\left(\mathbb{B}_m^T P_m + \mathbb{K}_m \mathbb{C}_m\right) - P_m \mathbb{B}_m \mathbb{B}_m^T P_m + \overline{Q}_m - \gamma^{-2} P_m A_{w,m} A_{w,m}^T P_m - \overline{A}_{h,m} \tilde{Q}_m^T \overline{A}_{h,m}^T \leq 0 \quad \text{(A26)}$$

Also, it can be rewritten as follows:

$$\mathbb{A}_m^T P_m + P_m \mathbb{A}_m + P_m \mathbb{B}_m \mathbb{K}_m \mathbb{C}_m + \left(\mathbb{B}_m^T P_m + \mathbb{K}_m \mathbb{C}_m\right)^T + \left(\mathbb{B}_m^T P_m + \mathbb{K}_m \mathbb{C}_m\right) - P_m \mathbb{B}_m \mathbb{B}_m^T P_m + \overline{Q}_m - \gamma^{-2} P_m A_{w,m} A_{w,m}^T P_m - \overline{A}_{h,m} \tilde{Q}_m^T \overline{A}_{h,}^T \leq 0 \quad \text{(A27)}$$

The equation (A28) is constructed to obtain the upper bound in inequality (A29):

$$[A_m + B_m K_m]_{ij} \geq 0 \quad 1 \leq i \neq j \leq n \quad \text{(A28)}$$

Then, it holds that:

$$[G_m - B_m K_m]_{ij} \geq 0 \quad 1 \leq i \neq j \leq n \quad \text{(A29)}$$

By substituting the upper bound of (A29) in (A27), we can get to:

$$\mathbb{A}_m^T P_m + P_m \mathbb{A}_m + P_m \mathbb{B}_m \mathbb{K}_m \mathbb{C}_m + \left(P_m \mathbb{B}_m + \mathbb{C}_m^T \mathbb{K}_m^T\right)\left(\mathbb{B}_m^T P_m + \mathbb{K}_m \mathbb{C}_m\right) - \varepsilon \mathbb{B}_m \mathbb{B}_m{}^T P_m - \varepsilon P_m \mathbb{B}_m \mathbb{B}_m{}^T + \varepsilon^2 \mathbb{B}_m \mathbb{B}_m{}^T + \overline{Q}_m - \gamma^{-2} P_m A_{w,m} A_{w,m}^T P_m - \overline{A}_{h,m} \tilde{Q}_m^T \overline{A}_{h,m}^T \leq 0 \quad \text{(A30)}$$

Now, applying Schur compliment to the previous relation leads to:

$$\begin{bmatrix} \theta_{11} & \overline{A}_{h,m} \\ * & \tilde{Q}_m \end{bmatrix} \leq 0$$

where

$$\begin{aligned} \theta_{11} = {} & \mathbb{A}_m^T P_m + P_m \mathbb{A}_m \\ & + \left(\mathbb{B}_m{}^T P_m + \mathbb{K}_m \mathbb{C}_m\right)^T \left(\mathbb{B}_m{}^T P_m + \mathbb{K}_m \mathbb{C}_m\right) \\ & - \varepsilon \mathbb{B}_m \mathbb{B}_m{}^T P_m - \varepsilon P_m \mathbb{B}_m \mathbb{B}_m{}^T + \varepsilon^2 \mathbb{B}_m \mathbb{B}_m{}^T \\ & + \overline{Q}_m - \gamma^{-2} P_m A_{w,m} A_{w,m}^T P_m \end{aligned} \quad \text{(A31)}$$

Applying Schur compliment two times:

$$\begin{bmatrix} \xi_{11} & \overline{A}_{h,m} & P_m \mathbb{B}_m + \mathbb{C}_m{}^T \mathbb{K}_m{}^T & \varepsilon \mathbb{B}_m \\ * & \tilde{Q}_m & 0 & 0 \\ * & * & -I & 0 \\ * & * & * & -I \end{bmatrix} \leq 0 \quad \text{(A32)}$$

where

$$\xi_{11} = \mathbb{A}_m^T P_m + P_m \mathbb{A}_m - \varepsilon \mathbb{B}_m \mathbb{B}_m{}^T P_m - \varepsilon P_m \mathbb{B}_m \mathbb{B}_m{}^T + \overline{Q}_m - \gamma^{-2} P_m A_{w,m} A_{w,m}^T P_m$$

By using one more Schur compliment on the term $\gamma^{-2} P_m A_w A_w^T P_m$:

$$\begin{bmatrix} \Gamma_m & \overline{A}_{h,m} & P_m \mathbb{B}_m + \mathbb{C}_m{}^T \mathbb{K}_m{}^T & \varepsilon \mathbb{B}_m & P_m A_{w,m} \\ * & \tilde{Q}_m & 0 & 0 & 0 \\ * & * & -I & 0 & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & -\gamma^2 I \end{bmatrix} \leq 0 \quad \text{(A33)}$$

where

$$\Gamma_m = \mathbb{A}_m^T P_m + P_m \mathbb{A}_m - \varepsilon \mathbb{B}_m \mathbb{B}_m{}^T P_m - \varepsilon P_m \mathbb{B}_m \mathbb{B}_m{}^T + \overline{Q}_m$$

With respect to (A15) and (A16) and integrating from t_k to t gives:

$$V(t) \leq e^{-\alpha(t-t_k)} V_{\sigma(t_k)}(t_k) - \int_{t_k}^{t} e^{-\alpha(t-s)} \left(z^T(s) z(s) - \gamma^2 \omega^T(s) \omega(s)\right) \mathrm{d}s \quad \text{(A34)}$$

Assume that $\sigma(t_k) = m_2$, $\sigma(t_k^-) = m_1$ at switching instant t_k, inequalities given in (20) guarantee

$$V_{m_2}(t_k) \leq \mu V_{m_2}(t_k^-) \quad \text{(A35)}$$

When $t \in [t_k, t_{k+1})$, from (A7) and (A9) we get to:

$$V_{\sigma(t)}(t) \leq e^{-\alpha(t-t_k)} V_{\sigma(t_k)}(t_k) \quad \text{(A36)}$$

And from Definition 3, after k times switching ($k \leq \frac{t-t_0}{\tau_a}$), it can be concluded that:

$$\begin{aligned} V_{\sigma(t)}(t) & \leq e^{-\alpha(t-t_k)} \mu V_{\sigma(t_k^-)}(t_k^-) \leq \\ & \cdots \leq e^{-\alpha(t-t_0)} \mu^k V_{\sigma(t_0)}(t_0) \\ & \leq e^{-(\alpha - \frac{ln\mu}{\tau_a})(t-t_0)} V_{\sigma(t_0)}(t_0) \end{aligned} \quad \text{(A37)}$$

We define $R_1 = max_m \lambda(P_m)$, $R_2 = max_m \lambda(Q_m)$ and $R_3 = min_m \lambda(P_m)$ thus:

$$\begin{aligned} V_{\sigma(t_0)}(t_0) & \leq \tilde{x}^T(t_0) R_1 \tilde{x}^T(t_0) + \int_{t_0-h(t_0)}^{t_0} \tilde{x}^T(t_0) R_2 \tilde{x}^T(t_0) \mathrm{d}s \\ & \leq R_1 \|\tilde{x}(t_0)\| + R_2 \overline{h} \|\tilde{x}(t_0)\| \end{aligned} \quad \text{(A38)}$$

It is derived from (A37) and (A38) that:

$$\begin{aligned} \tilde{x}^T(t) P_{\sigma(t)} \tilde{x}(t) & \leq V_{\sigma(t)}(t) \\ & \leq e^{-\left(\alpha - \frac{ln\mu}{\tau_a}\right)(t-t_0)} (R_1 + R_2 \overline{h}) \|\tilde{x}(t_0)\| \end{aligned} \quad \text{(A39)}$$

Consequently,

$$\|\tilde{x}(t)\| \leq e^{-\left(\alpha - \frac{ln\mu}{\tau_a}\right)(t-t_0)} (R_1/R_3 + R_2 \overline{h}/R_3) \|\tilde{x}(t_0)\| \quad \text{(A40)}$$

So, this bound $\tau_a > \frac{ln\mu}{\alpha}$ should be hold for ADT.

On the other hand, from (A34) and (A37) we get to:

$$\begin{aligned}
V(t) \le{} & e^{-\alpha(t-t_k)}\ V_{\sigma(t_k)}(t_k) \\
& - \int_{t_k}^{t} e^{-\alpha(t-s)}\left(z^T(s)\,z(s) - \gamma^2\omega^T(s)\,\omega(s)\right)\mathrm{d}s \\
& \le\ \mu^k\ V(t_0)\,e^{-\alpha t} \\
& - \mu^k\int_{t0}^{t1} e^{-\alpha(t-s)}\left(z^T(s)\,z(s) - \gamma^2\omega^T(s)\,\omega(s)\right)\mathrm{d}s \\
& - \mu^{k-1}\int_{t1}^{t2} e^{-\alpha(t-s)}\left(z^T(s)\,z(s) - \gamma^2\omega^T(s)\,\omega(s)\right)\mathrm{d}s - \\
& \cdots - \int_{tk}^{t} e^{-\alpha(t-t_k)}\left(z^T(s)\,z(s) - \gamma^2\omega^T(s)\,\omega(s)\right)\mathrm{d}s \\
={} & e^{-\alpha(t-t_0)+N_\sigma(t_0,t)ln\mu}V(t_0) \\
& - \int_{t0}^{t} e^{-\alpha(t-s)+N_\sigma(s,t)ln\mu}\Big(z^T(s)\,z(s) \\
& - \gamma^2\omega^T(s)\,\omega(s)\Big)\,\mathrm{d}s
\end{aligned} \tag{A41}$$

Under zero initial condition, (A41) gives:

$$0 \le - \int_{t0}^{t} e^{-\alpha(t-s)+N_\sigma(s,t)ln\mu}\left(z^T(s)\,z(s) - \gamma^2\omega^T(s)\,\omega(s)\right)\mathrm{d}s \tag{A42}$$

Simplifying the relation (A42) and integrating both sides of inequality from $t = t_0\ to\ \infty$ yields:

$$\int_0^\infty z^T(t)\,z(t)\,dt <\ \gamma^2\int_0^\infty \omega(t)^T\omega(t)\,\mathrm{d}t$$

H_∞ stabilization is guaranteed and the proof is complete.

Appendix III.

With respect to systems (26) and (27) we have:

$$\begin{aligned} e &= x(t) - \hat{x}(t) \\ &\Rightarrow \dot{x} = (A_m + B_mK_m)\,x(t) - B_mK_me(t) \end{aligned} \tag{A43}$$

$$\begin{aligned} \dot{e}(t) &= \dot{x}(t) - \dot{\hat{x}}(t) \\ &= (A_m + B_mK_m)\,x(t) - B_mK_me(t) \\ &- (G_m + L_mC_m)\,x(t) + G_me(t) \end{aligned} \tag{A44}$$

if $\tilde{x}(t) = \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}$ is considered, then we can rewrite to recent relations as:

$$\dot{\tilde{x}}(t) = \begin{bmatrix} A_m + B_mK_m & -B_mK_m \\ A_m + B_mK_m - G_m - L_mC_m & G_m - B_mK_m \end{bmatrix}\tilde{x}(t) \tag{A45}$$

By selecting the following Lyapunov function:

$$\begin{aligned} V_m(t) = V(t) &= \tilde{x}^T(t)P_m\tilde{x}(t) \\ &= x^T(t)P_{1,m}x(t) + e^T(t)P_{2,m}e(t) \end{aligned} \tag{A46}$$

The derivative of the Lyapunov function yields:

$$\begin{aligned} \dot{V} &= 2x^T(t)\,P_{1,m}\dot{x}(t) + 2e^T(t)\,P_{2,m}\dot{e}(t) \\ &= 2x^T(t)\,P_{1,m}(A_m + B_mK_m)\,x(t) - 2x^TP_{1,m}B_mK_me(t) \\ &+ 2e^T(t)\,P_{2,m}(A_m + B_mK_m - G_m - L_mC_m)\,x(t) \\ &+ 2e^T(t)\,P_{2,m}(G_m - B_mK_m)\,e(t) \\ &= \begin{bmatrix} x(t)^T & e(t)^T \end{bmatrix}\eta_m\begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \end{aligned} \tag{A47}$$

That η_m will be in this form:

$$\eta_m = \begin{bmatrix} P_{1,m}(A_m + B_mK_m) + (A_m + B_mK_m)^TP_{1,m} & -P_{1,m}B_mK_m + (A_m + B_mK_m - G_m - L_mC_m)^TP_{2,m} \\ -K_m^TB_m^TP_{1,m} + P_{2,m}(A_m + B_mK_m - G_m - L_mC_m) & P_{2,m}(G_m - B_mK_m) + (G_m - B_mK_m)^TP_{2,m} \end{bmatrix} < 0 \tag{A48}$$

For stability of the system, its necessary that $\eta_m < 0$, so in the following with respect to (32):

$$\mathbb{A}_m + \mathbb{B}_m\mathbb{K}_m = \begin{bmatrix} A_m + B_mK_m & -B_mK_m \\ A_m + B_mK_m - G_m - L_mC_m & G_m - B_mK_m \end{bmatrix} \tag{A49}$$

$$\eta_m = (\mathbb{A}_m + \mathbb{B}_m\mathbb{K}_m)^TP_m + P_m(\mathbb{A}_m + \mathbb{B}_m\mathbb{K}_m) < 0 \tag{A50}$$

It is also concluded that:

$$\eta_m + C_m^T\mathbb{K}_m^T\mathbb{K}_mC_m < 0 \implies \eta_m < 0 \tag{A51}$$

$$\begin{aligned} \eta_m + C_m^T\mathbb{K}_m^T\mathbb{K}_mC_m &= \mathbb{A}_m^TP_m + P_m\mathbb{A}_m \\ &+ \left(P_m\mathbb{B}_m + \mathbb{C}_m^T\mathbb{K}_m^T\right)\left(\mathbb{B}_m^TP_m + \mathbb{K}_m\mathbb{G}_m\right) \\ &- P_m\mathbb{B}_m\mathbb{B}_m^TP_m < 0 \end{aligned} \tag{A52}$$

Also, due to the (A28) and (A29) by placing the band above $-P_m\mathbb{B}_m\mathbb{B}_m^TP_m$ in (A52), we have:

$$\begin{aligned} &\mathbb{A}_m^TP_m + P_m\mathbb{A}_m + \left(P_m\mathbb{B}_m + \mathbb{C}_m^T\mathbb{K}_m^T\right)\left(\mathbb{B}_m^TP_m + \mathbb{K}_m\mathbb{G}_m\right) \\ &- \varepsilon\mathbb{B}_m{\mathbb{B}_m}^TP_m - \varepsilon P_m\mathbb{B}_m{\mathbb{B}_m}^T + \varepsilon^2\mathbb{B}_m{\mathbb{B}_m}^T < 0 \end{aligned} \tag{A53}$$

By applying Schur complement to the term $\left(P_m\mathbb{B}_m + \mathbb{C}_m^T\mathbb{K}_m^T\right)\left(\mathbb{B}_m^TP_m + \mathbb{K}_m\mathbb{G}_m\right)$:

$$\begin{bmatrix} \mathbb{A}_m^TP_m + P_m\mathbb{A}_m - \varepsilon\mathbb{B}_m{\mathbb{B}_m}^TP_m - \varepsilon P_m\mathbb{B}_m{\mathbb{B}_m}^T + \varepsilon^2\mathbb{B}_m{\mathbb{B}_m}^T & P_m\mathbb{B}_m + \mathbb{C}_m^T\mathbb{K}_m^T \\ * & -I \end{bmatrix} < 0 \tag{A54}$$

By applying Schur complement, one more this time to the term $\varepsilon^2\mathbb{B}_m{\mathbb{B}_m}^T$, we can reach to:

$$\Xi_m = \begin{bmatrix} \mathbb{A}_m^TP_m + P_m\mathbb{A}_m - \varepsilon\mathbb{B}_m\mathbb{B}_m^TP_m - \varepsilon P_m\mathbb{B}_m\mathbb{B}_m^T & P_m\mathbb{B}_m + G_m^TK_m^T & \varepsilon\mathbb{B}_m \\ * & -I & 0 \\ * & 0 & -I \end{bmatrix} < 0 \tag{A55}$$

On the other hand, Assume that $\sigma\left(t_k\right)=m_2,\ \sigma\left(t_k^-\right)=m_1, \mu \geq 1$ at switching instant t_k:

$$V_{m_2}\left(t_k\right) \leq \mu V_{m_2}\left(t_k^-\right) \tag{A56}$$

with respect to (A46) to (A50) and stability of the system, we assume $\lambda=-2\max_m \lambda\left(\mathbb{A}_m+\mathbb{B}_m \mathbb{K}_m\right)$, thus:

$$\dot{V}_{\sigma(t)}(t) \leq \lambda V_{\sigma(t)}(t) \tag{A57}$$

when $t \in [t_k, t_{k+1})$, from (A57) we get to:

$$V_{\sigma(t)}(t) \leq e^{-\lambda\left(t-t_k\right)}\ V_{\sigma\left(t_k\right)}\left(t_k\right) \tag{A58}$$

And from Definition 3, after k times switching ($k \leq \frac{t-t_0}{\tau_a}$), it can be concluded that:

$$V_{\sigma(t)}\left(t\right) \leq e^{-\lambda\left(t-t_k\right)}\ \mu V_{\sigma\left(t_k^-\right)}\left(t_k^-\right) \leq \ldots$$
$$\leq e^{-\lambda\left(t-t_0\right)} \mu^k\ V_{\sigma\left(t_0\right)}\left(t_0\right) \leq e^{-\left(\lambda-\frac{ln\mu}{\tau_a}\right)\left(t-t_0\right)} V_{\sigma\left(t_0\right)}\left(t_0\right) \tag{A59}$$

We define $R_1=\max_m \lambda\left(P_m\right)$,$R_2=\min_m \lambda\left(P_m\right)$ thus:

$$V_{\sigma\left(t_0\right)}\left(t_0\right) \leq \tilde{x}^T\left(t_0\right) R_1 \tilde{x}^T\left(t_0\right) \leq R_1\left\|\tilde{x}\left(t_0\right)\right\| \tag{A60}$$

It is derived from (A59) and (A58) that:

$$\tilde{x}^T\left(t\right) R_2 \tilde{x}(t) \leq\ V_{\sigma(t)}\left(t\right) \leq e^{-\left(\lambda-\frac{ln\mu}{\tau_a}\right)\left(t-t_0\right)} R_1\left\|\tilde{x}\left(t_0\right)\right\| \tag{A61}$$

Consequently,

$$\|\tilde{x}(t)\| \leq e^{-\left(\lambda-\frac{ln\mu}{\tau_a}\right)\left(t-t_0\right)}\left(R_1/R_2\right)\left\|\tilde{x}\left(t_0\right)\right\| \tag{A62}$$

So, this bound $\tau_a>\frac{\ln\mu}{\lambda}$ should be hold for ADT.

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